A Reference Governor Approach for Constrained Piecewise Affine Systems

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Abstract—In this paper we present an approach for designing of reference tracking controllers for constrained, discrete-time piecewise affine systems. The approach follows the idea of reference governor techniques where the desired set-point is filtered by a system called the “reference governor”. Based on the system current state, set-point, and prescribed constraints, the reference governor computes a new set-point for a low-level controller so that the state and input constraints are satisfied and convergence to the original set-point is guaranteed.

I. INTRODUCTION

Interest in control of piecewise affine (PWA) models stems from their capability of describing a large number of processes, such as: linear systems with static piecewise-linearities; linear systems with discrete states and inputs; switching systems where the dynamic behavior is described by a finite number of discrete-time linear models together with a set of logic rules for switching among these models; approximation of nonlinear dynamics, e.g., via multiple linearizations at different operating points.

In this work we focus on the design of reference tracking state-feedback controllers for constrained PWA systems. Our interest stems from industrial practice where, for control synthesis purposes, nonlinear plants are often approximated by partitioning the space spanned by the inputs, state, and exogenous signals into a finite number of regions (also called “modes”). Each region is then assigned an affine model and the nonlinear system is thus approximated by a PWA system. A standard gain scheduling strategy consists of designing a linear controller for each region along with an appropriate strategy for switching between them. In order to satisfy the state and input constraints, the control designer has to explicitly consider the case where a change in the reference signal results in a system transition between two or more regions.

In principle one could solve an optimal tracking problem for the constrained PWA systems by using the approach presented in [9]. There the authors have characterized the state-feedback solution to optimal control problems for PWA systems with performance criteria based on quadratic and linear norms. They have shown that the solution is a time-varying piecewise affine feedback control law, possibly defined over non-convex regions and proposed an algorithm that solves the Hamilton-Jacobi-Bellman equation by using a simple multiparametric solver. However, the implementation of the explicit controller might require significant computation infrastructure which might not be available for processes with fast sampling time and limited computational resources.

In this paper, we present an approach to the design of reference tracking controllers for constrained, discrete-time piecewise affine systems based on the concept of “reference governor” [3], [4], [18], [12]. Our approach is based on sets invariance theory and reachability analysis and, compared to the infinite time optimal solution [9], [8], is less computational demanding at the price of suboptimality and smaller region of attraction. The paper is organized as follows: Section II provides basic definitions on set invariance theory and reachability analysis. In Section III, we formulate the control problem, while in Section IV we present our main contribution, that is the reference governor approach. Section V closes the paper by illustrating the main ideas underlying the proposed approach and showing its effectiveness through a numerical example.

II. DEFINITIONS AND BASIC RESULTS

In this section we introduce a few definitions and then recall some basic results on multi-parametric programming and invariant set theory. We will denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{N}^+$, respectively.

A. Background on Invariant Sets

This section adopts the notation used in [10], [17], [11] and provides the basic definitions for invariant sets for constrained systems. A comprehensive survey of papers on set invariance theory can be found in [7].

Denote by $f_a$ the state update function of an autonomous system

$$x(t+1) = f_a(x(t))$$

subject to the constraints

$$x(t) \in \mathcal{X}$$

For the autonomous system (1)-(2), we denote the set of states that evolves to $\mathcal{S}$ in one step as

$$\text{Pref}_f(a)(\mathcal{S}) \triangleq \{x \in \mathcal{X} \mid f_a(x) \in \mathcal{S}\}$$

Equivalently, for the system with inputs

$$x(t+1) = f(x(t), u(t)),$$
subject to the constraints

\[ x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U}, \]  

(5)

the set of states which can be driven into the target set \( S \) in one time step is defined as

\[ \text{Pre}_f(S) \triangleq \{ x \in \mathcal{X} \mid \exists u \in \mathcal{U} \text{ s.t. } f(x, u) \in S \} \]  

(6)

The invariant sets are computed for autonomous systems and can be used to “find, for a given feedback controller \( u = k(x) \), the set of states whose trajectory will never violate the system constraints”. The following definitions are derived from [7], [6].

**Definition 1 (Positive Invariant Set):** A set \( O \) is said to be a positive invariant set for the autonomous system (1) subject to the constraints in (2), if

\[ x(0) \in O \quad \Rightarrow \quad x(t) \in O, \quad \forall t \in \mathbb{N}^+ \]

**Definition 2 (N-Step Stabilizable Set \( K_N(f;O) \)):** For a given invariant target set \( O \subseteq \mathcal{X} \), the \( N \)-step stabilizable set \( K_N(f;O) \) of the system (4) subject to the constraints (5) is defined as:

\[ K_N(f;O) \triangleq \text{Pre}_f(K_{N-1}(f;O)), \ N \in \mathbb{N}^+ \]

From Definition 2, all states \( x(0) \) belonging to the \( N \)-step stabilizable set \( K_N(f;O) \) can be driven, through a time-varying control law, to the target set \( O \) in \( N \) steps and stay in \( O \) for all \( t \geq N \) while satisfying input and state constraints.

An equivalent definition can be given for the autonomous system (1) subject to the constraints in (2); all states \( x(0) \) belonging to the \( N \)-Step Stabilizable Set \( K_N(f;O) \) will reach to the target set \( O \) in \( N \) steps and stay in \( O \) for all \( t \geq N \) while satisfying state constraints.

### III. PROBLEM FORMULATION

Consider the PWA system

\[ x(t + 1) = A_i x(t) + B_i u(t) + f_i \]

if \( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{P}_i \), \( i = \{1, \ldots, s\} \),

(7)

where \( x \in \mathbb{R}^n \), \( u \in \mathbb{R}^m \), \( \{\mathcal{P}_i\}_{i=1}^s \) is a polyhedral partition of the set of the state and input space \( \mathcal{P} \subseteq \mathbb{R}^{n+m} \). The current index \( i \) will be called the system mode, i.e., the PWA system (7) is in mode \( i \) at time \( t \) if \( \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{P}_i \).

System (7) is subject to hard input and state constraints

\[ x(t) \in \mathcal{X}, \ u(t) \in \mathcal{U} \]

(8)

for \( t \geq 0 \), and we denote by Constrained PWA system (CPWA) the restriction of the PWA system (7) over the set of states and inputs defined by (8),

\[ x(t + 1) = A_i x(t) + B_i u(t) + f_i \quad \text{if} \quad \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{P}_i \],

(9)

where \( \{\mathcal{P}_i\}_{i=1}^s \) is the new polyhedral partition of the sets of state and input space \( \mathbb{R}^{n+m} \) obtained by intersecting the sets \( \mathcal{P}_i \) in (7) with the polyhedron described by (8). We assume the following.

**Assumption 1:** For a given reference state \( x_{\text{ref}} \) there is an unique input \( u_{\text{ref}} = u_{\text{ref}}(x_{\text{ref}}) \) such that \( x_{\text{ref}} = A_i x_{\text{ref}} + B_i u_{\text{ref}} + f_i \) if \( \begin{bmatrix} x_{\text{ref}} \\ u_{\text{ref}} \end{bmatrix} \in \mathcal{P}_i \).

The function \( u_{\text{ref}}(x_{\text{ref}}) \) is unique either from the properties of system (9) (there is one mode and one \( u_{\text{ref}} \) for each \( x_{\text{ref}} \)) or by construction (i.e., for the given \( x_{\text{ref}} \) the user specifies the desired mode and the corresponding \( u_{\text{ref}} \)).

**Remark 1:** Assumptions 1 is introduced for the sake of simplicity and it is not restrictive. It could be easily removed at the cost of a more complex notation. In particular, if multiple equilibria are allowed, the best \( u_{\text{ref}} \) is typically chosen as a result of an optimization problem [15].

Our objective is to design a state feedback control law \( u(x, x_{\text{ref}}) \) such that the closed loop system

\[ x(t + 1) = A_i x(t) + B_i u(x(t), x_{\text{ref}}) + f_i \quad \text{if} \quad \begin{bmatrix} x(t) \\ u(x(t), x_{\text{ref}}) \end{bmatrix} \in \mathcal{P}_i \]

(10)

converges to \( x_{\text{ref}} \) and satisfies state and input constraints.

A systematic approach to design constrained reference tracking controllers is to use a receding horizon control policy. We define the following cost function

\[ J_N(U_N, x(0), x_{\text{ref}}) \triangleq \| x_{\text{ref}} - x(t) \|^2_P + \sum_{k=0}^{N-1} \| x_k - x_{\text{ref}} \|^2_P + \| u_k - u_{\text{ref}}(x_{\text{ref}}) \|^2_P \]

(11)

with \( Q = Q' \geq 0, R = R' > 0, P \geq 0, \| x \|^2_M \) and consider the constrained finite-time optimal control (CFTOC) problem

\[ J_0^*(x(0), x_{\text{ref}}) \triangleq \min_{U_N} J(U_N, x(0), x_{\text{ref}}) \]

\[ x_{k+1} = A_i x_k + B_i u_k + f_i \quad \text{if} \quad \begin{bmatrix} x_k \\ u_k \end{bmatrix} \in \mathcal{P}_i, \ i = 1, \ldots, s \]

sub. to

\[ x_{ref,k+1} = x_{ref,k} \]

\[ k = 0, \ldots, N - 1 \]

\[ \begin{bmatrix} x_N, x_{\text{ref}} \end{bmatrix} \in \mathcal{X}_f \]

\[ x_0 = x(0), \ x_{\text{ref},0} = x_{\text{ref}} \]

where the column vector \( U_N \triangleq [u_0, \ldots, u_{N-1}] \in \mathbb{R}^{nN} \) is the optimization vector, \( N \) is the optimal control horizon. \( \mathcal{X}_f \) is a polyhedral terminal region in the \([x, x_{\text{ref}}]-\)space. Note that we distinguish between the input \( u(t) \) and the state \( x(t) \) of plant (9) at time \( t \) and the variables \( u_k \) and \( x_k \) of the optimization problem (13).

We will also denote by \( \mathcal{X}_k \subseteq \mathbb{R}^2 \) the set of states \( x_k \) and references \( x_{\text{ref}} \) that are feasible for (11)-(13):

\[ \mathcal{X}_k \triangleq \left\{ x \in \mathbb{R}^n, \begin{bmatrix} u \in \mathbb{R}^m, \exists i \in \{1, \ldots, s\} \end{bmatrix} \begin{bmatrix} x \in \mathcal{P}_i \end{bmatrix} \right\} \]

\[ x_{\text{ref}} \in \mathcal{R} \]

\[ k = 0, \ldots, N - 1 \]

\[ \mathcal{X}_N = \mathcal{X}_f \]

(14)

Note that the optimizer function \( U_N^* \) may not be uniquely defined if the optimal set of problem (11)-(13) is not a singleton for some \( x(0) \). The next theorem shows the properties of the optimal control solution.
Theorem 1 ([9]): Consider the optimal control problem (11)-(13). Then, there exists a solution in the form of a PW A system (9). The proposed control law is simply obtained by repeatedly evaluating at each time $t$ the PW controller (15) for $k = 0$:  

$$u(t) = u_0(x(t), x_{ref}) \text{ for } \left[ \begin{array}{l} x(t) \\ x_{ref} \end{array} \right] \in \tilde{X}_0.$$  

(16) 

If $\tilde{X}_f$ is a control invariant set and the terminal cost $P$ is a control Lyapunov function, then for all $[x(0), x_{ref}] \in \tilde{X}_0$ the system state $x(k)$ will converge to the desired constant reference $x_{ref}$ while satisfying input and state constraints [13]. Note that the sets $\tilde{X}_k$ are defined in the state and reference space, since the initial state and the reference are both external parameters of the optimal control problem (11)-(13). 

The number of regions in the solution to (16) might prohibit the real-time implementation for systems with limited computational and storage resources. In the next section we propose an alternative approach based on the results presented in [3], [4], [5], [2], [1] and show how to design a low-complexity controller which guarantees constraint satisfaction by using the idea of reference governor.

IV. REFERENCE GOVERNOR

Consider the constrained PW A system (9). The proposed control design approach is based on the following three main steps. First, local tracking controllers are designed for each mode $i$ of the PW system and the invariant sets $O^i$, in the state and reference space, are computed for the corresponding closed loop systems. Second, for any pair of modes $(i, j)$, transition controllers are designed for steering the current state in mode $i$ to the invariant set $O^j$ in mode $j$. Last, for any pair of modes $(i, j)$, an optimal sequence of transitions is computed from the mode $i$ to the mode $j$ as the shortest path on a weighted graph. The graph weights are functions of the “transition cost” between any two modes. The online reference governor algorithm solves a simple constrained Quadratic Programming (QP) problem in order to modify the reference and move to the next mode according to the determined shortest path. The three steps are detailed next.

1) Computation of local tracking controllers

For each mode $i$ design a state-feedback controller $k^i(x, x_{ref})$ for tracking the reference $x_{ref}$ in mode $i$. Denote by $O^i$ the positive invariant set of the corresponding closed loop system. $O^i$ is a polyhedron in the $[x, x_{ref}]$ space such that if at time $k$ $[x(k), x_{ref}] \in O^i$, then state and input constraints (8) are satisfied for all $t \geq k$ and $x(t) \rightarrow x_{ref}$.

2) Computation of transition controllers

For each pair $i, j$ of modes, design a transition controller $k^{ij}(x, x_{ref})$. Then, for the closed loop system $f_{ij}^{x, u}$, compute the corresponding $N^{ij}$-step stabilizable set $X^{ij} = K_{N^{ij}}((P^i)^{ij}, O^j)$, i.e., the set of states and references in mode $i$ which are steered by $k^{ij}(x, x_{ref})$ to the invariant set $O^j$ in mode $j$ in at most $N^{ij}$ steps. $X^{ij}$ is a union of polyhedra in the $[x, x_{ref}]$ space. Denote by $X^j_{x_{ref}}$ the projection of $X^{ij}$ on the $x$ space, i.e., $X^j_{x_{ref}} = \text{Proj}_x(X^{ij})$. If $x$ belongs to $X^j_{x_{ref}}$ then there exists a (time-varying) reference which steers the PW A system (7) from mode $i$ to the invariant in mode $j$ while satisfying state and input constraints. In particular, if at time $k$ there exists a $x_{ref}(k)$ such that $[x(k), x_{ref}(k)] \in X^{ij}$, then there exists $p \leq N^{ij}$ and a reference trajectory $x_{ref}(k), \ldots, x_{ref}(k+p)$ such that $[x(k+p), x_{ref}(k+p)] \in O^j$ under the control law $k^{ij}(x(k), x_{ref}(k)), \ldots, k^{ij}(x(k+p-1), x_{ref}(k+p-1))$.

3) Computation of optimal switching policy

For each pair $i, j$ of modes, compute the best sequence of transitions $\{i, i_1, i_2, \ldots, i_p, j\}$ from mode $i$ to mode $j$ as the shortest path on a weighted graph. The nodes of the graph represent the system modes and the weights on the arcs represent the cost of switching between two adjacent modes.

Algorithm 4.1:

**Input:** Current state $x$ and reference $x_{ref}$

**Output:** Modified reference $\tilde{x}_{ref}$ and control input $u(x, \tilde{x}_{ref})$

1) Read $x$ and $x_{ref}$ and let $r$ be the mode of the reference: $[x_{ref}, u_{ref}] \in O^r$.

2) If $[x_{ref}, x_{ref}] \notin O^r$, $x_{ref}$ is an infeasible reference, EXIT.

3) Else if $[x, x_{ref}] \notin O^r$ then set $\tilde{x}_{ref} = x_{ref}$ and apply $u = k^i(x, \tilde{x}_{ref})$.

4) Else if $x_{ref}$ exists such that $[x, x_{ref}] \in O^r$ then choose $\tilde{x}_{ref}$ with $[x, \tilde{x}_{ref}] \in O^r$ “close” to $x_{ref}$ and apply $u = k^j(x, \tilde{x}_{ref})$.

5) Else Let $V = \{v_1, \ldots, v_p\}$ be the set of modes to which $x$ can be steered to, i.e., $\exists j \in \{1, \ldots, n\}, \exists v_k \in V$ such that $x \in X^k_{x_{ref}}$. Note that $j$ might depend on the applied input.

6) Compute the mode $v^* \in V$ with associated minimum cost to reach $r$. If there exists no path from any $v_k \in V$, then the problem is infeasible, EXIT.

7) Compute $\tilde{x}_{ref}$ such that $[x, \tilde{x}_{ref}] \in X^{i^*, v^*}$ and apply $u = k^{i^*v^*}(x, \tilde{x}_{ref})$.

8) Go to Step 1.
online reference governor ensures that the system converges to the desired reference while satisfying input and state constraints. In order to help the reader to understand the main idea, a simplified description of the on-line reference governor is given by Algorithm 4.1. The algorithm will be detailed later in Section IV-D.

Note that Assumption 1 has been used in Step 1 of the Algorithm where it is assumed that \(w_{ref}(x_{ref})\) is unique.

### A. Local Control design

For each region \(\tilde{P}_i\), the following reference tracking controller is considered

\[
u = k^i(x, x_{ref}) \tag{17}
\]

where \(k^i(x, x_{ref})\) is a stabilizing control law. For each region \(\tilde{P}_i\) we compute a positive invariant set \(O^i\) for the closed loop system:

\[
x_{k+1} = A^i x_k + B^i k^i(x_k, x_{ref,k}) + f^i,
\]

\[
x_{ref,k+1} = x_{ref,k},
\]

subject to the constraints

\[
\begin{bmatrix}
x_k \\
\tilde{k}^i (x_k, x_{ref, k})
\end{bmatrix} \in \tilde{P}_i, \quad \begin{bmatrix}
x_{ref, k} \\
\hat{u}_{ref, k}
\end{bmatrix} \in \tilde{P}_i. \tag{19}
\]

We remark that \(O^i\) is a set in the \([x, \tilde{x}_{ref}]\) space. We assume that \(k^i\) guarantees the convergence of \(x(k)\) to a constant reference \(x_{ref}\) for system (18).

Assume \([\hat{u}_{ref}, \tilde{x}_{ref}]\) \in \(\tilde{P}_i\). If \([x_{ref}, \tilde{x}_{ref}]\) \in \(O^i\) then the controller \(k^i(x, x_{ref})\) will (i) guarantee constraint satisfaction at all time instants, (ii) keep the system in mode \(i\) and (iii) guarantee convergence to \([\hat{u}_{ref}, \tilde{x}_{ref}]\) (step 3 of the Online Algorithm). If \([x_{ref}, \tilde{x}_{ref}]\) \notin \(O^i\) then the local controller \(k^i\) will not guarantee feasibility and will not drive \(x_0\) towards \(x_{ref}\). However, two cases are possible:

1. \(\hat{x}_{ref}\) might exist such that \([\hat{x}_{ref}, \hat{x}_{ref}]\) \in \(O^i\).
2. \(\hat{u}\) exists such that \([\hat{u}, \hat{u}]\) \in \(P^i\) and a “transition controller” \(k^i(x, \tilde{x}_{ref})\) that steers the system from mode \(i\) to mode \(i\) through a modified \(\tilde{x}_{ref}\).

In the both cases feasibility can be guaranteed by computing a new reference \(x_{ref}\). The second case requires “transition controllers”. The design of such controllers is described next.

### B. Transition Control Design

For each \((i, j), i \neq j\), select an horizon \(N_{i,j}\). For a given linear or PWA transition controller \(k^{i,j}(x, x_{ref})\), denote by \(f_{i,j}^{k}\) the closed loop PWA system in region \(i\), i.e.,

\[
\begin{bmatrix}
x_{k+1} \\
\tilde{x}_{ref, k+1}
\end{bmatrix} = f_{i,j}^{k}(x_k, x_{ref, k}) \triangleq \begin{bmatrix}
A^{i} x_k + B^{i} k^{i,j}(x_k, x_{ref, k}) + f^i \\
\tilde{k}^{i,j} (x_k, x_{ref, k})
\end{bmatrix}
\]

and by \(\mathcal{X}_{i,j}\) the set of states and references which are steered from mode \(i\) to the set \(O^i\) in mode \(j\) in at most \(N_{i,j}\) steps, i.e., \(\mathcal{X}_{i,j} \triangleq K_{N_{i,j}}(f_{i,j}^{k}, O^i)\). Note that if \([x_{ref}, \tilde{x}_{ref}]\) \notin \(\mathcal{X}_{i,j}\), then the reference \(x_{ref}\) can be modified to \(\hat{x}_{ref}\) in order to have \([x_{ref}, \tilde{x}_{ref}]\) \in \(\mathcal{X}_{i,j}\) and steer the system to mode \(j\) by using the controller \(k^{i,j}\). Clearly \(\mathcal{X}_{i,j}\) might be empty.

Recall that \(\mathcal{X}_{i,j}^{ej}\) is the projection of \(\mathcal{X}_{i,j}\) on the \(x\) space. Similarly, \(\mathcal{X}_{i,j}^{ej}\) is the projection of \(\mathcal{X}_{i,j}^{ej}\) on the \(x\) space. If \(x\) belongs to \(\mathcal{X}_{i,j}^{ej}\) then there exist a reference which will bring the PWA system from mode \(i\) to the invariant set \(O^j\) in mode \(j\) in \(p\) steps.

### C. The Weighted Graph

For each mode we have designed a local controller \(k^i\) and computed a corresponding invariant \(O^i\). For each pair of modes we have designed a transition controller \(k^{i,j}\) and computed a set \(\mathcal{X}_{i,j}\) of states and references in mode \(i\) which reach \(O^j\) in mode \(j\) in at most \(N_{i,j}\) steps. Clearly, if the current state is in mode \(i_1\) and the reference in mode \(i_2\), the system could still be controlled to the reference even if \(\mathcal{X}_{i_1,j_0}\) is empty. Therefore, the last step is to compute the optimal transition sequence \(i_1, i_2, \ldots, i_n\) between any two modes \(i_1\) and \(i_2\). We propose to use the properties of the sets \(O^i\) and \(\mathcal{X}_{i,j}\) in order to avoid the inherent exponential complexity of the problem at the price of smaller regions of attraction. In particular we can move from \(i_1\) to \(i_2\) through the set \(\mathcal{X}_{i_1,j_2}\) by using \(k^{i_1,j_2}\). Then, when the system is in mode \(i_2\), we can move through the set \(O^{j_2}\) by using \(k^{j_2}\) and reach \(\mathcal{X}_{i_2,j_3}\) and so on. In this way input and state constraints are always satisfied. The feasibility property of this approach is described in following proposition.

**Proposition 1**: Let Assumption 1 hold, \(x(k)\) be the current system state and \(x_{ref}\) the current reference. Assume that \([x_{ref}, u_{ref}(x_{ref})]\) is in mode \(j\). Assume \(x(k) \in \text{Proj}_j(O^i)\). Define the set \(\mathcal{X}_{i,j}\) as

\[
\mathcal{X}_{i,j} \triangleq \text{Proj}_j(O^i) \cap \text{Proj}_j \{[x, x_{ref}] \in \mathcal{X}_{i,j} | x = x_{ref}\} \tag{20}
\]

If \(\mathcal{X}_{i,j}\) is not empty, then there exists a time varying reference and a feasible state feedback control law such that the system (9) with initial state \(x(k)\) in mode \(i\) can be steered to the set \(O^j\) in mode \(j\).

**Proof**

We consider two cases: \(x(k) \in \text{Proj}_j(\mathcal{X}_{i,j})\) and \(x(k) \notin \text{Proj}_j(\mathcal{X}_{i,j})\).

If \(x(k) \in \text{Proj}_j(\mathcal{X}_{i,j})\) then compute \(\hat{x}_{ref}\) such that \([x(k), \hat{x}_{ref}] \in \mathcal{X}_{i,j}\) and apply \(k^{i,j}(x_k, \hat{x}_{ref})\). By construction of \(\mathcal{X}_{i,j}\), there exists a sequence of references such that the system will reach \(O^j\) in at most \(N_{i,j}\) steps.

Assume \(x(k) \notin \text{Proj}_j(\mathcal{X}_{i,j})\). By assumption \(x(k) \in \text{Proj}_j(O^i)\), and \(\mathcal{X}_{i,j}\) is not empty. Therefore the system can stay in \(O^i\) (and thus satisfy constraints) and by changing the reference it can reach a new state \(\hat{x}_{ref}\) with the property \(\hat{x}_{ref} \in \text{Proj}_j(\mathcal{X}_{i,j})\).

Pick \([\hat{x}_{ref}, \hat{\tilde{x}}_{ref}] \in \mathcal{X}_{i,j}\) with \(\hat{x}_{ref} \in \mathcal{X}_{i,j}\) and solve the following problem

\[
\hat{x}_{ref} \triangleq \arg \min_{\hat{x}_{ref}} (\|\hat{x}_{ref} - \tilde{x}_{ref}(\hat{x}_{ref})\|) \tag{21a}
\]

subject to \([x(k), \hat{x}_{ref}] \in \mathcal{X}_{i,j}\). \(\hat{x}_{ref}\) is feasible since by assumption \(x(k) \in \text{Proj}_j(O^i)\). Note however that \([x(k), \hat{x}_{ref}]\) might not belong to \(O^i\). Apply \(k^{i,j}(x, \hat{x}_{ref})\). Since \([x(k), \hat{x}_{ref}] \in \mathcal{X}_{i,j}\) then \(\lim_{k \to +\infty} x(k) = \hat{x}_{ref}\) and since \(O^i\) is a connected set, \(\lim_{k \to +\infty} \tilde{x}_{ref} = \hat{x}_{ref}\) with \([\hat{x}_{ref}, \hat{x}_{ref}] \in \mathcal{X}_{i,j}\). \(\square\)
Proposition 1 shows that we can transition from mode $i$ to mode $j$ from $X_{i,j}$ or from $O_i$ if $\bar{X}_{i,j}$ is not empty. Therefore, one can steer the state from mode $i_1$ to mode $i_n$ by applying the sequence of controllers $k_1, k_2, k_3, \ldots, k_n$.

The concept of weighted graph will be used to compute the "best" transition sequence from $O_i$ to $O_j$ for any two modes $i$, $j$. A weighted graph $G$ is defined as

$$G = (V, A)$$

(22)

where $V$ is the set of nodes (or vertices) $V = \{1, \ldots, N\}$ and $A \subseteq V \times V$ the set of edges $(i, j)$ with $i \in V, j \in V$. Let $A_{i,j} \in \mathbb{R}$ be the $i,j$ element of the weighted adjacency matrix $A$ of the graph $G$. If there is no edge connecting the vertex $i$ with the vertex $j$, i.e., $(i, j) \notin A$, we set $A_{i,j} = 0$.

The elements of $A$ are computed as follows:

$$a_{i,j} = \alpha \frac{1}{\text{vol}(X^{i,j})} + \beta \delta_{i,j}$$

(23)

where $\text{vol}(P)$ is the volume of the polyhedron $P$. The positive real numbers $\alpha$ and $\beta$ are tuning parameters. Given the weighted graph $G$, $u = \text{SPath}(G, i_1, i_n)$ is the vector which describes the shortest path $u = \{i_1, i_2, \ldots, i_n\}$ between node $i_1$ and node $i_n$ and $\text{SPathCost}(G, i_1, i_n)$ is the corresponding optimal cost.

Clearly, several other alternatives to the weight choice in (23) can be proposed depending on the specific application. The weights in (23) capture only two important elements: the time to reach the target region and the size of the feasible set which generates a transition. The latter can be seen as a practical measure on how robust to system uncertainties and measurement noise the transition is.

D. On-line Reference Governor Algorithm

Once all the elements have been computed off-line, the following algorithm is implemented on-line.

Algorithm 4.2:

Input: Current state $x(t)$ and reference $x_{ref} = x_{ref}(t)$
Output: Modified reference $\hat{x}_{ref}(t)$ and controller selection
1. let $r$ be such that $[x_{ref}, x_{ref}] \in O^r$
2. if $x(t) \in \text{Proj}_x(O^r)$ then select local controller $k^r$ and compute the modified reference as follows
   $$\hat{x}_{ref} = \arg \min_{\hat{x}_{ref}} \|\hat{x}_{ref} - x_{ref}\|$$
   subj. to $[\hat{x}(t)] \in O^r$
(24a)
(24b)
3. else let $v = \{v_1, \ldots, v_n\}$ the set of modes such that $x(t) \in \text{Proj}_x(X_1, v_1)$ and let $u = \{u_1, \ldots, u_n\}$ the set of modes such that $x(t) \in \text{Proj}_x(O_{u_1})$. (note that $x(t)$ can be in multiple modes because the system partition depends on the input as well).
4. Compute $v^* \in v \cup u$ with the associated shortest path $\{v^*, i_1, \ldots, i_n, r\}$ and cost $s^* = \text{SPathCost}(G, v^*, r)$.
5. if $s^* = \infty$ then “Infeasible Reference”, EXIT
6. if $v^* \in v$ then select transition controller $k^{v^*, v^*}$ and compute the modified reference as follows
   $$\hat{x}_{ref} = \arg \min_{\hat{x}_{ref}} \|\hat{x}_{ref} - x_{ref}\|$$
   subj. to $[\hat{x}(t)] \in X^{v^*, v^*}$
(25)
(26)
8. else select local controller $k^{v^*}$ and compute the modified reference as follows
   $$\hat{x}_{ref} = \arg \min_{\hat{x}_{ref}} \|\hat{x}_{ref} - x_{ref}\|$$
   subj. to $[\hat{x}(t)] \in X^{v^*, i_1}$
(27)
(28)
9. go to Step 1

Remark 2: Note that the sets $X^{i,j}$ might be described as the union of polyhedra $A_i^{k,j}$, for $k = 1, \ldots, N^{i,j}$ where $A_i^{k,j}$ represents the $k$-steps reachable set. In this case, Step 7 in Algorithm 4.2 can be modified as follows:

$$\hat{x}_{ref} = \arg \min_{\hat{x}_{ref}, k} \|\hat{x}_{ref} - x_{ref}\|$$

subj. to $[\hat{x}(t)] \in X^{k,v^*, v^*}$
(30a)
(30b)

The same modification can be applied to Step 8 in Algorithm 4.2.

Remark 3: Note that the QP problem defined in Step 2 in Algorithm 4.2 can be solved explicitly as has been shown in [16], [14].

Remark 4: Consider step 5 of Algorithm 4.2. If there exists multiple $v^*$ yielding $s^* = \text{SPathCost}(G, v^*, r)$, then a $v^*$ will be used. In this case cycling could occur but it can easily be avoided by storing modes which have been already explored.

V. NUMERICAL EXAMPLE

Next we present a simple numerical example for the algorithm presented in Section IV. We consider the second
order PWA system (31).

\[
x(t + 1) = 0.8 \begin{bmatrix} \cos \alpha(t) & -\sin \alpha(t) \\ \sin \alpha(t) & \cos \alpha(t) \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

\[
\alpha(t) = \begin{cases} 
-\frac{\pi}{4} & \text{if } [0, 1]x(t) < 1 \\
\frac{\pi}{4} & \text{if } [0, 1]x(t) \geq 1
\end{cases}
\]

\[
x(t) \in [-10, 50] \times [-10, 10] \quad u(t) \in [-2, 1]
\]

(31)

The control objective is to bring the state of the system (31) to the final state \(x_{ref} = [-1.4846, -1.2857]') in mode \(\tilde{P}_1^1\) with associated \(u_{ref} = -1.800\).

In each region \(\tilde{P}_i^1, i = 1, 2\), the system (31) is controlled by a state feedback linear controller \(k_i(x, x_{ref}) = k_i(x - x_{ref})\) with \(k_1 = [-0.4214, 0.3112]\) and \(k_2 = [0.4214, 0.3112]\). For the sake of simplicity, we focus only on the following transition: steer the state of the system (31) from mode 2 to mode 1. We select \(N^{12} = 15\) and compute the set \(\mathcal{X}^{21}\), that is, the subset of \(\tilde{P}_2^2\) which can be steered in at most 15 steps into \(\mathcal{O}_\infty^{1}\). \(\mathcal{X}^{21}\) is composed of five polyhedra and the projection \(\mathcal{X}_0^{21}\) is depicted in Figure 1(b).

If \(x(t) \in \text{Proj}_x(\mathcal{X}^{21})\), Algorithm 4.2 solves the QP problem (30) to bring system in mode 1 by applying the feedback control \(k_i^2(x, x_{ref})\). Once the state of the system (31) is in \(\mathcal{O}_\infty^1\), the linear control law \(k_1(x, x_{ref})\) is used to control the system to the state \([x_{ref}]\) with \(x_{ref}\) being the solution to the QP (24). Simulation of the closed-loop system from \(x(0) = [\bar{x}_0]\) are depicted in Figure 1(a).

We observe that after \(t = 4\) the system is in mode 1 and controlled to the state \(x_{ref}\), while the control input satisfies the input constraints.

REFERENCES


