On a Property of a Class of Offset-Free Model Predictive Controllers

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Abstract—We consider a model predictive control framework that includes a discrete-time linear time-invariant nominal plant model augmented with an output integrator disturbance model and a Kalman filter to estimate the state and disturbance vectors. While the application of this framework can guarantee offset-free control, it has shown a consistent limitation in the achievable closed loop estimator performance. Using root locus techniques, we identify sufficient conditions for a class of nominal plant models with at least one real pole for which the closed loop estimator poles cannot be arbitrarily selected regardless of the augmented system’s statistics. We present several examples illustrating the limitations of the closed loop estimator pole locations.

I. INTRODUCTION

The main concept of Model Predictive Control (MPC) is to use a model of the plant to predict the future evolution of the system [6], [11]. At each time step a certain performance index is optimized over a sequence of future input moves subject to operating constraints. The first of such optimal moves is the control action applied to the plant at time . At time , a new optimization is solved over a shifted prediction horizon.

Steady-state offset refers to asymptotically constant biases between the controlled output vector and the steady-state reference vector. MPC algorithms are designed to achieve offset-free steady-state tracking by augmenting the plant model with a disturbance model. This disturbance model is used to predict the bias error between the measured output vector and the output vector predicted using the nominal plant model. The general approach of these offset-free MPC algorithms is, first, to estimate the state and disturbance vectors using the measured output vector and, second, to use the estimated state and disturbance vectors to initialize the MPC optimization problem.

A number of disturbance models have been proposed and applied in offset-free MPC algorithms for linear and nonlinear nominal plant models [4], [9], [10], [12], [13], [15]–[17], [19]. These disturbance models consist of integrating modes and are selected to capture the type of uncertainty affecting the nominal plant model. In [13] and [17], observability conditions on the augmented system model were derived to ensure steady-state offset-free tracking for generalized disturbance models where the uncertainty can enter the nominal plant model through the state vector, input vector, or output vector. However, even if the augmented system model satisfies such observability criteria, the authors in [1], [12], [13], [17] have observed a consistent limitation in the achievable closed loop system performance when Kalman filters are applied to nominal systems models augmented with output integrator disturbance models.

In this paper, we consider the following widely used offset-free MPC framework: (i) a discrete linear time-invariant nominal plant model, (ii) an output integrator disturbance model, and (iii) a linear constant-gain Kalman filter. The objective of this paper is to present and study the source of the limitations of closed loop estimator performance for this offset-free MPC framework. In particular, we prove that if nominal plant models with at least one real pole satisfy certain conditions, then the resulting closed loop estimator has one real pole that cannot be arbitrarily selected. Furthermore, this limitation can be independent of the statistics of the nominal plant and disturbance models.

This paper is organized as follows. In Section II, we describe the nominal plant model and output integrator disturbance model. In Section III, we review the constant-gain Kalman filter and formulate the relationship between the closed loop estimator poles and the nominal plant model poles. In Section IV, we prove the limitations on the placement of the closed loop estimator poles for the defined offset-free MPC framework. In Section V, we present several examples illustrating the limitations described in Section IV.

II. PROBLEM FORMULATION

We consider the following nominal linear time-invariant (LTI) system:

where \( x_k \in \mathbb{R}^n \) and \( u_k \in \mathbb{R}^m \) are the system state and input vectors at time , respectively, \( z_k \in \mathbb{R}^p \) is both the measurement vector and the controlled output vector, \( w_{x,k} \in \mathbb{R}^n \) and \( v_k \in \mathbb{R}^p \) are the state process noise and measurement noise vectors, respectively, and are modeled as zero-mean, Gaussian, white sequence uncorrelated with both \( w_{x,k} \) and \( v_k \):

\[
E \left[ w_{x,k} w_{x,k}^T \right] = Q_x = Q_x^T \geq 0, \quad E \left[ v_k v_k^T \right] = R = R^T > 0, \\
E \left[ w_{x,i} v_j^T \right] = 0 \forall i, j, k,
\]
where $Q_\sigma \in \mathbb{R}^{n \times n}$ is the state process noise covariance matrix and $R \in \mathbb{R}^{p \times p}$ is the measurement noise covariance matrix. For analysis throughout the paper, we rewrite $R$ as $R = r\bar{R}$ where $\bar{R} > 0$ and $r \in \mathbb{R}$, $r > 0$. We assume that $(A, B)$ is stabilizable, $(A, G_x\sqrt{Q_x})$ is stabilizable where $Q_x = \sqrt{Q_x}Q_x\sqrt{Q_x}$, and $(C, A)$ is observable.

The objective of controller design is to formulate a control law that enables the controlled output vector to track an asymptotically constant reference vector $z_{ref}$, where $z_{ref} \in \mathbb{R}^p$. To achieve steady-state offset-free tracking of $z_{ref}$, we augment the nominal system (1) with an integrator disturbance model:

$$
x_{k+1} = Ax_k + Bu_k + B_d d_k + G_x w_{x,k},
$$

$$
d_{k+1} = d_k + G_d w_{d,k},
$$

$$
z_k = Cx_k + C_d d_k + v_k,
$$

where $d_k \in \mathbb{R}^{n_d}$ is the disturbance vector and $w_{d,k} \in \mathbb{R}^{n_d}$ is the disturbance process noise vector at time $k$. The vector $w_{d,k}$ is modeled as a zero-mean, Gaussian, white sequence uncorrelated with $v_k$:

$$
E \left[ w_{d,k} w_{d,k}^T \right] = Q_d > 0, \forall k, \quad G_d Q_d G_d^T > 0,
$$

$$
E \left[ w_{x,i} w_{d,j}^T \right] = 0, \forall i, j,
$$

where $Q_d \in \mathbb{R}^{n_d \times n_d}$ is the disturbance process noise covariance matrix. We assume that the state and input constraints are inactive at steady-state throughout this paper.

Remark 1: We refer the reader to [13] for a description of the augmented system (3).

We denote $I_n$ as the identity matrix belonging to $\mathbb{R}^{n \times n}$ and assume the following:

Assumption 1: We consider the output integrator disturbance model with $n_d = p$, $C_d = I_p$, and $B_d = 0$ in (3).

The augmented system (3) with Assumption 1 can be compactly written as:

$$
X_{k+1} = \Phi X_k + \bar{B}u_k + \Gamma w_k,
$$

$$
z_k = HX_k + v_k,
$$

where $X_k \in \mathbb{R}^{n+n_d}$ is the augmented state vector at time $k$:

$$
X_k = \left[ \begin{array}{c} x_k^T \\ d_k^T \end{array} \right],
$$

$w_k \in \mathbb{R}^{n+n_d}$ is the augmented process noise vector:

$$
w_k = \left[ \begin{array}{c} w_{x,k}^T \\ w_{d,k}^T \end{array} \right],
$$

$$
E \left[ w_k w_k^T \right] = Q \triangleq \left[ \begin{array}{cc} Q_x & 0 \\ 0 & Q_d \end{array} \right],
$$

and

$$
\Phi = \left[ \begin{array}{cc} A & 0 \\ 0 & I_{n_d} \end{array} \right], \quad \bar{B} = \left[ \begin{array}{c} B \\ 0 \end{array} \right], \quad \Gamma = \left[ \begin{array}{cc} G_x & 0 \\ 0 & G_d \end{array} \right],
$$

$$
H = \left[ \begin{array}{cc} C & I_p \end{array} \right].
$$

We assume that $(\Phi, \Gamma \sqrt{Q})$ is stabilizable where $Q = \sqrt{Q_x}Q_x\sqrt{Q_x}$ and $(H, \Phi)$ is observable. Necessary and sufficient conditions for the observability of the augmented system (5) are given in the following proposition.

**Proposition 1:** The augmented system (5) is observable if and only if $(C, A)$ is observable and

$$
\left[ \begin{array}{cc} A - I_n & 0 \\ C & I_p \end{array} \right]
$$

has full column rank. (9)

**Proof:** See [13], [17].

Remark 2: In Assumption 1, we require $n_d = p$. In [17] it was proven that this condition guarantees steady-state offset-free tracking of $z_{ref}$. We refer the reader to [17] for a description of offset-free tracking for the case $n_d < p$.

Remark 3: To satisfy condition (9) of Proposition 1, the nominal system cannot have integrating modes.

### III. Kalman Filter Design and Multivariable Root Locus

This section describes the constant-gain Kalman filter used to estimate the augmented state vector governed by (5) and the frequency domain relationship between the steady-state closed loop estimator poles and the nominal system poles. The term steady-state will be dropped when describing the closed loop estimator poles because the following analysis applies at steady-state.

We use a Kalman filter [8] to estimate the augmented state vector. A constant-gain Kalman filter exists for the augmented system (5) because the matrices $\Phi$, $H$, $Q$, and $R$ are time-invariant, $Q \geq 0$, $R > 0$, $(\Phi, \Gamma \sqrt{Q})$ is stabilizable, and $(H, \Phi)$ is observable. The closed loop estimator equations can be written as [8]:

$$
\dot{\hat{X}}_{k+1} = \Phi(\hat{X}_n + n_d) - KH\hat{X}_k + \bar{B}u_k + \Phi K \hat{z}_k,
$$

$$
K = P_{\infty}H^T(HP_{\infty}H^T + R)^{-1},
$$

$$
P_{\infty} = \Phi P_{\infty} \Phi^T - \Phi P_{\infty} H^T(HP_{\infty}H^T + R)^{-1}HP_{\infty} \Phi^T + \Gamma \Gamma^T,
$$

where $K$ is the Kalman gain, $P_{\infty} \geq 0$ is the solution to the discrete algebraic Riccati equation, $(H P_{\infty} H^T + R)$ is nonsingular, and $|\lambda_i(\Phi(\hat{X}_n + n_d) - KH\hat{X}_k)| < 1$ for all $i = 1, \ldots, n + n_d$ where $\lambda_i(M)$ denotes the $i$-th eigenvalue of the matrix $M$.

For a constant-gain Kalman filter implemented for an LTI system, the frequency domain can be used to analyze the relationship between the closed loop estimator poles and the nominal system poles. We first define the following three transfer functions for analysis purposes. We denote $\Delta(z)$ as the characteristic polynomial of the augmented system (5):

$$
\Delta(z) = |zI_{n+n_d} - \Phi| = |zI_n - A||zI_{n_d} - I_{n_d}| = |zI_n - A|(z - 1)^{n_d},
$$

where $|.|$ refers to the determinant of $(|.)$. We denote $\Delta_{cl}(z)$ as the characteristic polynomial of the closed loop estimator (10):

$$
\Delta_{cl}(z) = |zI_{n+n_d} - \Phi(\hat{X}_n + n_d) - KH\hat{X}_k| = |zI_{n+n_d} + \Phi KH(zI_{n+n_d} - \Phi)^{-1}||zI_{n+n_d} - \Phi| = |zI_{n+n_d} + H(zI_{n+n_d} - \Phi)^{-1}HK|\Delta(z).
$$

We denote $G(z)$ as the transfer function of the augmented system from $\hat{w}_k$ to $\hat{z}_k$ where $\hat{w}_k = \sqrt{Q}\hat{w}_k$ and $\hat{w}_k$ is a white noise vector with unit covariance matrix:

$$
G(z) = H(zI_{n+n_d} - \Phi)^{-1}\sqrt{Q} = \frac{H|\hat{w}_k|}{|zI_{n+n_d} - \Phi|} \triangleq \frac{N_G(z)}{\Delta(z)}.
$$

(13)
\[ \Delta_{cl}(z) \Delta_{cl}(z^{-1}) = |H(zI_n + n_d - \Phi)^{-1} \Gamma Q \Gamma^T (z^{-1} I_n + n_d - \Phi)^{-T} H^T + R| \cdot \Delta(z) \Delta(z^{-1}), \]  

(14)

\[ \Delta_{cl}(z) \Delta_{cl}(z^{-1}) = \frac{\text{Adj}(zI_n - A) G x \text{Adj}(z^{-1} I_n - A^T) C^T}{|zI_n - A||z^{-1} I_n - A^T|} + \frac{G d Q_d G_d^T}{(z - 1)(z^{-1} - 1)} + R| \Delta(z) \Delta(z^{-1}), \]  

(15)

\[ \Delta_{cl}(z) \Delta_{cl}(z^{-1}) = |-z^{-1} n(z) N^T (z^{-1}) (z-1)^2 + G_d Q_d G_d^T | zI_n - A | z^{-1} I_n - A^T | + R | zI_n - A||z^{-1} I_n - A^T | (z - 1)(z^{-1} - 1)| \frac{1}{|zI_n - A||z^{-1} I_n - A^T|.} \]  

(16)

The Chang-Letov equation [8] can be used to establish the relationship between \( \Delta(z) \), \( \Delta_{cl}(z) \), and \( G(z) \) and analyze the locations of the steady-state closed loop estimator poles:

\[ \Delta_{cl}(z) \Delta_{cl}(z^{-1}) = |H(zI_n + n_d - \Phi)^{-1} \Gamma Q \Gamma^T (z^{-1} I_n + n_d - \Phi)^{-T} H^T + R| \cdot \Delta(z) \Delta(z^{-1}) \]

(17)

where \( |H P \infty H^T + R| \) is a scaling factor and will be ignored in the following analysis. If (13) is substituted into (17), then \( \Delta_{cl}(z) \Delta_{cl}(z^{-1}) \) can be rewritten as:

\[ \Delta_{cl}(z) \Delta_{cl}(z^{-1}) = |N_c(z) N_c^T (z^{-1}) + R| \cdot \Delta(z) \Delta(z^{-1}) = (18a) \]

\[ = |N_c(z) N_c^T (z^{-1}) + R \Delta(z) \Delta(z^{-1}) | \frac{1}{(z - 1)(z^{-1} - 1)}. \]  

(18b)

The right hand side of (18) is a polynomial whose stable roots are the closed loop estimator poles. Note that in (18b) the term \( \frac{1}{\Delta(z) \Delta(z^{-1})} \) simplifies with the polynomial resulting from the computation of \( |N_c(z) N_c^T (z^{-1}) + R \Delta(z) \Delta(z^{-1})| \).

The Chang-Letov equation (18) can be used to analyze the location of the closed loop estimator poles. As \( R \to 0 \), the \( N_c(z) N_c^T (z^{-1}) \) term predominates. Therefore, the closed loop estimator poles approach the stable zeros of \( G(z) G^T (z^{-1}) \) when \( p = n_d = 1 \). As \( R \to \infty \), the \( \Delta(z) \Delta(z^{-1}) \) term predominates. Therefore, the closed loop estimator poles approach the stable zeros of \( \Delta(z) \Delta(z^{-1}) \) which are the stable poles of \( G(z) G^T (z^{-1}) \). In other words, the closed loop estimator poles approach either the augmented system’s stable poles or the augmented system’s unstable poles reflected inside the unit circle because the closed loop estimator is stable. Unstable poles are reflected inside the unit circle by taking their inverse.

The polynomial \( \Delta_{cl}(z) \Delta_{cl}(z^{-1}) \) can be written explicitly as a function of the augmented state matrices by substituting (8) into (17). The resulting equation (14) is written at the top of the page. By explicating the inverse of the matrix in (14), we obtain (15) which can be compactly written as (16) where

\[ N(z) = \text{Adj}(zI_n - A) G x \sqrt{Q}_{x}. \]  

(19)

Equation (16) is written at the top of the page.

A. Multivariable Root Loci

Root locus techniques can be used to analyze the loci of the closed loop estimator poles as \( 0 \to R \to \infty \) and, thus, we provide a brief overview of multivariable root loci properties. The following definitions and results have been extracted from [14], [18], [21], [22] and are based on a generic MIMO transfer function \( F(z) \).

Definition 1: [22] Let the Smith-McMillan form [7] of \( F(z) \) be \( \text{diag}(\frac{n_i(z)}{d(z)}) \), let \( p \) be a pole of order \( n \) of \( F(z) \), and let \( z \) be a zero of order \( n \) of \( F(z) \). \( p \) is simple if the polynomial \( (z - p)^n \) divides \( d_1(z) \) exactly and \( d_i(p) \neq 0 \) for all \( i > 1 \). \( z \) is simple if the polynomial \( (z - z)^n \) divides \( n_m(z) \) exactly and \( n_i(z) \neq 0 \) for all \( i < m \). We denote \( F_{pp}(z) = d_1 \cdot d_m \) as the pole polynomial of \( F(z) \).

Definition 2: \( F(z) \) is regular if (i) its real poles and zeros are simple, (ii) the leading matrix coefficients of the Laurent series expansions of \( F(z) \) at any simple pole and \( F(z)^{-1} \) at any simple zero have rank which equals the number of their non-zero eigenvalues (simple null structure), and (iii) there are no single-point loci on the real axis.

Note that in [5] it was proven that there is little loss in generality by assuming that the Laurent series expansions of \( F(z) \) and \( F(z)^{-1} \) have simple null structures.

Definition 3: Let \( F(z) \) be a polynomial such that \( \Phi(z, \kappa) \triangleq F_p(z)(I + \kappa F(z)) \) where the gain \( K = \kappa I \) and \( \kappa \in \mathbb{R}, \kappa > 0 \). The point \( s_0 \) is defined as a branch point at the gain \( \kappa_0 \) if \( \Phi(s_0, \kappa_0) = 0 \) and \( \frac{\partial \Phi}{\partial \kappa}(s_0, \kappa_0) = 0 \).

Theorem 1: [22] Consider a regular MIMO transfer function \( F(z) \) and the loci of its \( n \) closed-loop poles corresponding to the one parameter family of gains \( K = \kappa I \). The number, \( N_{r_n} \), of branches of the root loci of \( F(z) \) at a point \( s_0 \) on the real axis is given by:

\[ N_{r_n} = \sum \text{poles } p_i \text{ of odd order } n_i \text{ sign } (z - p_i)^{n_i} \text{ trace } F(z)_z = p_i + \sum \text{number of asymptotes at } \pm \infty + \sum \text{zeros } z_i \text{ of odd order } n_i \text{ sign } (z - z_i)^{n_i} \text{ trace } F(z)^{-1}_{z = z_i} + 2 \sum \text{branch points } b_i \text{ of odd order } n_i \text{ sign } \frac{\partial^2 \Phi}{\partial \kappa^2} \frac{\partial \Phi}{\partial \kappa} |_{z = b_i} \]  

(20)

where \( n_i \) are the orders of the poles and zeros.

IV. MAIN RESULT

We have often observed a limitation in the closed loop estimator performance when using the Kalman filter (10) to estimate the augmented state vector governed by (5). This section identifies and describes this performance characteristic by considering the root loci of the closed loop estimator poles. We consider nominal systems with at least one real pole.

Assumption 2: There exists an \( i \) such that \( \lambda_i(A) \in \mathbb{R} \).

We define the following quantities for analysis purposes and begin by classifying real eigenvalues of the nominal system. We denote \( \Lambda_r \) as the set of real eigenvalues of \( A \) and...
their inverses, \( \Lambda_r = \{ \lambda_i, 1/\lambda_i \mid \lambda_i \in \mathbb{R}, i = 1, \ldots, n \} \). We denote \( \lambda_{\max,r} \) as the largest stable eigenvalue in \( \Lambda_r \), \( \lambda_{\max,r} = \max_{|\lambda| < 1} \lambda \). We denote \( \lambda_{\min,r} \) as the smallest stable eigenvalue in \( \Lambda_r \), \( \lambda_{\min,r} = \min_{|\lambda| < 1} \lambda \). We denote \( \lambda_{\min,r,2} \) as the second smallest stable eigenvalue in \( \Lambda_r \), \( \lambda_{\min,r,2} = \min_{|\lambda| < 1, \lambda \neq \lambda_{\min,r,2}} \{ \lambda \in \Lambda_r, + \infty \} \). Finally, we denote \( Z(z) = \Delta_{cl}(\lambda) \Delta_{cl}(z^{-1}) \) from (16) when \( r = 0 \). \( Z(z) \) is given in (21) at the top of the page.

In the following, we consider classes of both single output and multiple output augmented systems with \( \lambda_{\max,r} > 0 \) and \( \lambda_{\min,r} < 0 \). For these systems, we provide general results regarding the root loci of the closed loop estimator poles, i.e., the stable roots of \( \Delta_{cl}(\lambda) \Delta_{cl}(z^{-1}) \) as \( 0 \to r \to \infty \). In particular we show that there are branches of the root loci that lie on the real axis and that these branches correspond to either slow or fast closed loop estimator poles which are independent of \( r \). The existence of these branches is related to the assumption that the stable roots of \( Z(z) \) lie within certain segments of the real axis. For each class of augmented system, we provide sufficient conditions on \( Z(z) \) that guarantee the existence of such roots. We first consider systems with \( \lambda_{\max,r} > 0 \) [2].

**Theorem 2:** Consider the class of augmented systems (5) with \( p = 1 \) that satisfy Assumption 2. If \( \lambda_{\max,r} > 0 \) and the following condition is satisfied:

\[
\exists \bar{\lambda} \in \mathbb{R} \mid \lambda_{\max,r} < \bar{\lambda} < 1 \text{ and } Z(\bar{\lambda}) = 0,
\]

then for all \( r > 0 \), there exists \( \lambda_{cl} \in \mathbb{R} \) such that \( \Delta_{cl}(\lambda_{cl}) = 0 \) and \( \lambda_{\max,r} \leq \lambda_{cl} \leq 1 \).

**Proof:** As discussed in Section III, the branches of the root locus of \( \Delta_{cl}(\lambda) \Delta_{cl}(z^{-1}) \) move from the zeros of \( G(z)G^T(z^{-1}) \) to the poles of \( G(z)G^T(z^{-1}) \) as \( 0 \to r \to \infty \). These branches have the following properties: (1) they must begin at a zero (which can include \( \infty \)) and end at a pole, (2) a real branch lies to the left of an odd number of real poles and zeros [20], (3) the branches are symmetric about the real axis, (4) the branches are inversely symmetric about the unit circle because if \( g(z) = 0 \) is a branch, then \( g(z^{-1}) = 0 \) is a branch, and (5) a branch can not cross the unit circle because the closed loop estimator is stable. \( G(z)G^T(z^{-1}) \) has exactly two poles at \( z = 1 \) (one pole from the disturbance model, one pole from its inverse, and no other poles at \( z = 1 \) from Remark 3). Property 1 indicates that two branches of the root locus must end at \( z = 1 \). Properties 4 and 5 indicate that one branch lies inside the unit circle and the second branch lies outside the unit circle. Property 3 indicates that these two branches must lie on the real axis. Property 2 indicates that the stable branch must begin at the largest zero of \( G(z)G^T(z^{-1}) \), \( \bar{\lambda}_{\max} \), that satisfies condition (22). Therefore, there is a stable, real branch of the root locus of \( \Delta_{cl}(\lambda) \Delta_{cl}(z^{-1}) \) that begins at \( \bar{\lambda}_{\max} \) and ends at \( z = 1 \).

**Proposition 2:** If \( p = 1 \), \( \lambda_{\max,r} > 0 \), and

\[
N(\lambda_{\max,r})N^T(\lambda_{\max,r}) > 0,
\]

then there exists a \( \bar{z} \) satisfying condition (22) in Theorem 2.

**Proof:** Evaluate the right hand side of \( Z(z) \) in (21) at \( z = \lambda_{\max,r} \) and \( z = 1 \):

\[
Z(z)_{z=\lambda_{\max,r}} = -\lambda_{\max,r}^1N(\lambda_{\max,r})N^T(\lambda_{\max,r})|\lambda_{\max,r}^{-1}|^2, \quad (24)
\]

\[
Z(z)_{z=1} = G_dQ_dG_d^T[I_n - A]|I_n - A|. \quad (25)
\]

Equation (24) and hypothesis (23) imply that \( Z(\lambda_{\max,r}) < 0 \). Equation (25) implies that \( Z(1) > 0 \) because \( G_dQ_dG_d^T > 0 \) from (4) and \( |I_n - A| > 0 \) from Remark 3. The polynomial \( Z(z) \) is continuous for all \( z > 0 \) and, thus, it must have a real root between \( z = \lambda_{\max,r} \) and \( z = 1 \).

By using the definitions given in Section III-A and Theorem 1, the results of Theorem 2 can be extended to multiple output nominal systems.

**Theorem 3:** Consider the class of augmented systems (5) with \( p > 1 \) that satisfy Assumption 2. If \( \lambda_{\max,r} > 0 \) and the following conditions are satisfied:

1. the function \( Z(z) \) is regular,
2. \( \exists \bar{z} \in \mathbb{R} \mid \lambda_{\max,r} < \bar{z} < 1 \text{ and } Z(\bar{z}) = 0 \), (26)
3. let \( \bar{z}_{\max} \triangleq \max \{ \bar{z} \text{ satisfying (26)} \} \) with multiplicity \( k \). Assume \( k \) is odd and:
4. \[
\text{sign} \left( |z - \bar{z}_{\max}|^k \text{trace} Z^{-1}(z) \right)_{z=\bar{z}_{\max}} = -1, \quad (27)
\]
5. \( Z(z) \) has no real branch points \( z_0 \) such that \( \bar{z}_{\max} \leq z_0 < 1 \), then, for all \( r > 0 \), there exists \( \lambda_{cl} \in \mathbb{R} \) such that \( \Delta_{cl}(\lambda_{cl}) = 0 \) and \( \lambda_{\max,r} \leq \lambda_{cl} \leq 1 \).

**Proof:** Condition (MO1) allows us to apply Theorem 1 to count the branches on the real axis to the left and right of \( \bar{z}_{\max} \) defined in (MO3). Condition (MO3) and Theorem 1 ensure that there will be one less real branch to the left of \( \bar{z}_{\max} \) as compared to the right of \( \bar{z}_{\max} \). Therefore, there will be a real branch that begins at \( \bar{z}_{\max} \) and moves to the right of \( \bar{z}_{\max} \). Condition (MO4) ensures that this branch will not double back on itself and retrace its path. Condition (MO2) ensures that \( Z(z) \) has no roots between \( \bar{z}_{\max} \) and \( z = 1 \). Therefore, there is a stable, real branch of the root loci of \( \Delta_{cl}(\lambda) \Delta_{cl}(z^{-1}) \) that begins at \( \bar{z}_{\max} \) and ends at \( z = 1 \).

**Proposition 3:** If \( p > 1 \), \( \lambda_{\max,r} > 0 \), and

\[
Z(z)_{z=\lambda_{\max,r}} < 0, \quad (28)
\]

then there exists a \( \bar{z} \) satisfying condition (26) in Theorem 3.

**Proof:** Evaluate the right hand side of \( Z(z) \) in (21) at \( z = \lambda_{\max,r} \) as in (29) and at \( z = 1 \):

\[
Z(z)_{z=1} = |G_dQ_dG_d^T[I_n - A]|I_n - A|. \quad (30)
\]

Equation (29) and hypothesis (28) imply that \( Z(\lambda_{\max,r}) < 0 \). Equation (30) implies that \( Z(1) > 0 \) because \( G_dQ_dG_d^T > 0 \).
Z(z)|_{z=\lambda_{\text{max},r}} = \frac{(-1)^p \left| -z^{-1}N(z)N^T(z^{-1})(z-1)^2 + G_dQ_dG_d^T \right|_{z=\lambda_{\text{max},r}}}{\left| zI_n - A \right|^{p-1}\left| zI_n - A \right|^{p-1}} \tag{29}

from (4) and $|I_n - A| > 0$ from Remark 3. The polynomial $Z(z)$ is continuous for all $z > 0$ and, thus, it must have a real root between $z = \lambda_{\text{max},r}$ and $z = 1$. \hfill \Box

Remark 4: Evaluating (28) is not straightforward because $Z(z)|_{z=\lambda_{\text{max},r}} \neq (-1)^p \left| -zN(z)N^T(z^{-1})(z-1)^2 \right|_{z=\lambda_{\text{max},r}}$ as in the single output case. It should be noted that Proposition 2 is sufficient for Theorem 2 to hold whereas Proposition 3 is sufficient only for one condition (26) of the four conditions required for Theorem 3 to hold.

The following two theorems extend the results of Theorems 2 and 3 to systems with $\lambda_{\text{min},r} < 0$. We denote $\hat{\xi}_{\text{min}}$ as the smallest real zero of $Z(z)$ between $z = \lambda_{\text{min},r}$ and $z = 0$:

$$\hat{\xi}_{\text{min}} = \min \{ \xi \in \mathbb{R} \mid \xi \leq 0 \text{ and } Z(\xi) = 0 \}.$$ \tag{32}

Equation (16) indicates that $\Delta_d(z)\Delta_d(z^{-1})$ will always have at least one zero at $z = 0$ when $R$ is zero because as $r \to 0$ at least one root locus branch will move towards infinity and the corresponding branch reflected inside the unit circle will move toward zero. Therefore, the maximum value of $\hat{\xi}_{\text{min}}$ is $z = 0$.

Theorem 4: Consider the class of augmented systems (5) with $p = 1$ that satisfy Assumption 2. If $\lambda_{\text{min},r} < 0$ with odd multiplicity $k$ and the following condition is satisfied

$$\lambda_{\text{min},r} < \hat{\xi}_{\text{min}} < \lambda_{\text{min},r,2},$$ \tag{33}

then, for all $r > 0$, there exists $\lambda_{d} \in \mathbb{R}$ such that $\Delta_d(\lambda_{d}) = 0$ and $\lambda_{\text{min},r} \leq \lambda_{d} \leq 0$.

Proof: The branches of the root locus of $\Delta_d(z)\Delta_d(z^{-1})$ must satisfy the five properties given in the proof of Theorem 2. Since $\lambda_{\text{min},r}$ has odd multiplicity, property 2 ensures that a real branch arrives at $\lambda_{\text{min},r}$ from either its right or left. Condition (33) and property 5 ensure that this branch arrives from the right of $\lambda_{\text{min},r}$ because there are no real stable zeros to the left of $\lambda_{\text{min},r}$ and the branch cannot cross the unit circle. Condition (33) and property 1 ensure that this branch begins at $\hat{\xi}_{\text{min}}$. Therefore, there is a stable, real branch of the root locus of $\Delta_d(z)\Delta_d(z^{-1})$ that begins at $\hat{\xi}_{\text{min}}$ and ends at $z = \lambda_{\text{min},r}$. \hfill \Box

Theorem 5: Consider the class of augmented systems (5) with $p > 1$ that satisfy Assumption 2. If $\lambda_{\text{min},r} < 0$ with odd multiplicity $k$, $\hat{\xi}_{\text{min}}$ has odd multiplicity $j$, and the following conditions are satisfied:

(MO1) the function $Z(z)$ is regular,

(MO2) $\text{sign} \left( (z - \lambda_{\text{min},r})^k \text{trace} \left. Z^{-1}(z) \right|_{z=\lambda_{\text{min},r}} \right) = -1,$ \tag{34}

(MO3) $\lambda_{\text{min},r} < \hat{\xi}_{\text{min}} < \lambda_{\text{min},r,2}$ and

$$\text{sign} \left( (z - \hat{\xi}_{\text{min}})^j \text{trace} \left. Z^{-1}(z) \right|_{z=\hat{\xi}_{\text{min}}} \right) = 1.$$ \tag{35}

V. EXAMPLES

Example 1: Single State, Single Output Nominal Model.
Consider the following augmented system (5):

$$X_{k+1} = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix} X_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w_k,$$

$$z_k = [c \ 1] X_k + v_k$$ \tag{36}
where $\lambda, c, Q_x, Q_d, R = r \in \mathbb{R}$, $\Delta(z) = (z - \lambda)(z - 1)$, and $N(z) = c \, \text{adj}(z - \lambda)\sqrt{Q_x} = c \sqrt{Q_x}$.

Consider the case $\lambda > 0$. We denote the stable eigenvalue in $\Lambda_r$ as $a = \text{min} \{\lambda, 1/\lambda\}$. Therefore, $\|N(a)N^T(a^{-1})\| = c^2Q_x > 0$ for all $Q_x > 0$. From Proposition 2 and Theorem 2 we can conclude that the closed loop estimator always has a real pole between $z = a$ and $z = 1$ for all $r > 0$. In particular, the closed loop estimator will have a stable, real branch of the root locus of $\Delta_\ell(z)\Delta_\ell(z^{-1})$ that begins at $a < \hat{z} < 1$ and ends at $z = 1$ where $\hat{z} = \text{min}\{z_1, z_1^{-1}\}$ and:

$$ z_1 = \frac{2c^2Q_x + (1 + a^2)Q_d}{2c^2Q_x + 2a_0Q_d} + \left[\frac{a^2Q_x + (a + 1)^2Q_d}{2c^2Q_x + 2aQ_d}\right]^{1/2}. \quad (37) $$

Therefore, the closed loop estimator always has a non-oscillating mode slower than $z = a$.

Consider the case $\lambda < 0$. If $c^2Q_x - aQ_d > 0$, then $z_1 > 0$, $z_1^{-1} > 0$, and $\hat{z}_{\min} = 0$ in Theorem 4. The root locus plot is shown in case 2 of Figure ???. If $c^2Q_x - aQ_d < 0$, then $a < z_1 < 0$ and $\hat{z}_{\min} = z_1$ in Theorem 4. From Theorem 4 we can conclude that the closed loop estimator always has a real pole between $z = a$ and $z = 0$. In particular, the closed loop estimator will have a stable, real branch of the root locus of $\Delta_\ell(z)\Delta_\ell(z^{-1})$ that begins at $\hat{z}_{\min}$ and ends at $z = a$. Therefore, the closed loop estimator always has an oscillating mode faster than $z = a$.

Additional examples can be found in [2], [3].

VI. CONCLUSION

In this paper, we considered an offset-free MPC framework with a discrete LTI nominal system model, an output integrator disturbance model, and a constant-gain Kalman filter. We identified sufficient conditions for a class of systems with at least one real pole for which the closed loop estimator pole locations could not be arbitrarily selected. In particular, using root locus techniques, we showed that the closed loop estimator always has a non-oscillating mode slower than the slowest, stable, positive nominal system real mode and an oscillating mode faster than the slowest, stable, negative nominal system real mode. These limitations on the closed loop estimator poles restrict the locations of the closed loop MPC poles and, thus, restrict the closed loop controller performance.

The limitations on the closed loop estimator pole locations are a result of using the Kalman filter for the defined offset-free MPC framework. However, two other approaches can be considered that might not lead to these limitations. First, an alternative observer design method can be used with the defined offset-free MPC framework. For example, in [16], the observer gain matrix was designed using $H_\infty$ techniques and limitations on closed loop estimator pole locations were not reported. Second, the Kalman filter can be used with the defined offset-free MPC framework with correlated state and disturbance process noise vectors. In [19], it was shown that the closed loop performance of a nominal plant model augmented with an output integrator disturbance model with correlated process noise vectors was equivalent to the performance of a nominal plant model augmented with a standard input integrator disturbance model. Therefore, by relaxing the assumption on uncorrelated state and disturbance process noise vectors, one might ease or eliminate the restrictions on the closed loop estimator pole locations.

REFERENCES